Wave attractors in anisotropic media

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1 density-stratified fluids, $\boldsymbol{g}$

2 rotating fluids, $\boldsymbol{\Omega}$

3 plasma's, $\boldsymbol{B}$

4 metamaterial, $\boldsymbol{\varepsilon}$

Isotropic (2D) surface gravity waves:

Velocity potential $\varphi \propto e^{i(kx+ly-\omega t)-kz} \rightarrow$ $\omega^2 = gk\tanh(kH)$, $\boldsymbol{k} = (k,l) = k(\cos\alpha, \sin\alpha)$

No constraint on *direction*! $\omega = \omega(\kappa) \rightarrow \boldsymbol{c}_g = \nabla_{\boldsymbol{k}}\omega \parallel \boldsymbol{c} = \frac{\omega}{k^2}\mathbf{k}$

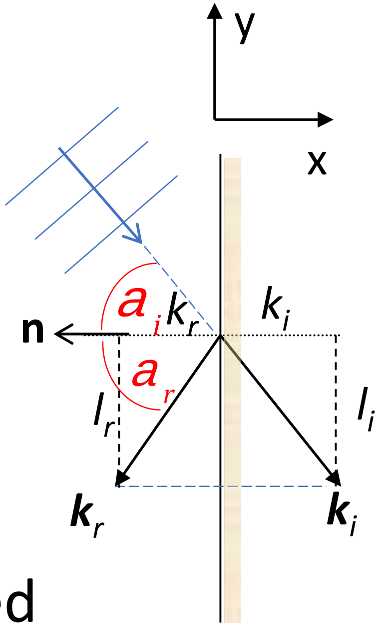
$\omega_r = \omega_i \rightarrow k_r = k_i$

$\boldsymbol{u} = (u,v) = \nabla\varphi = i\boldsymbol{k}\varphi = i(k,l)\varphi$

$(\boldsymbol{u}_i + \boldsymbol{u}_r)\cdot\boldsymbol{n} = 0$

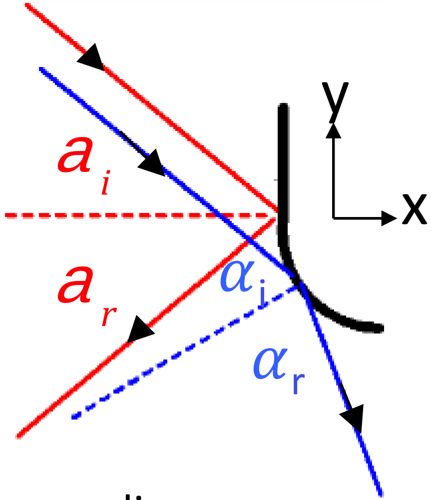
$(\boldsymbol{k}_i + \boldsymbol{k}_r)\cdot\boldsymbol{n} = 0 \rightarrow k_r = -k_i$

continuity of $u_\parallel$

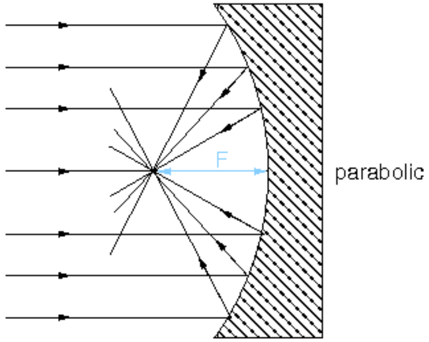


Wave length conserved

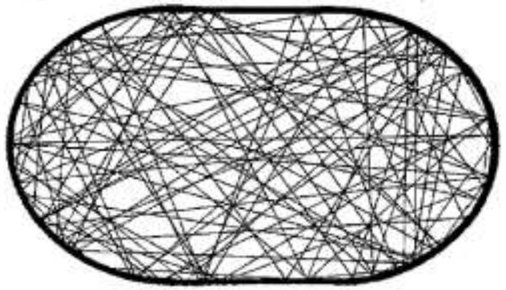
$\alpha_r = \alpha_i$ specular reflection



Wave ray divergence



Transient focusing



Berry 1987, 'billiard dynamics': Ray chaos

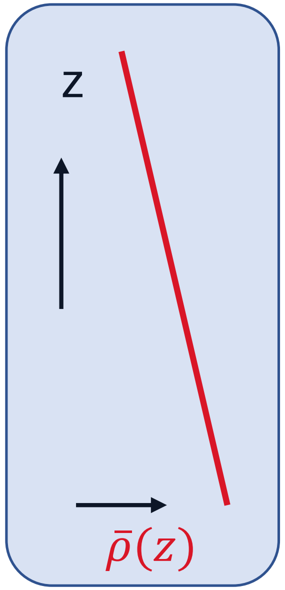
Anisotropic fluid

Uniform stratification $N = \sqrt{-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}}$ =constant

Heat & Salt => Density:

$r = r_0 + \bar{r}(z) + r'(x,z,t),$

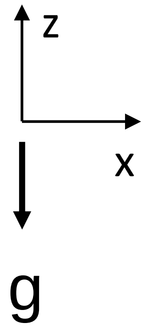
$r_0 \gg \max(\bar{r}(z)) \gg \max(r'(x,z,t))$



$\frac{1}{2} < \frac{\omega}{N} < 1$

Side view

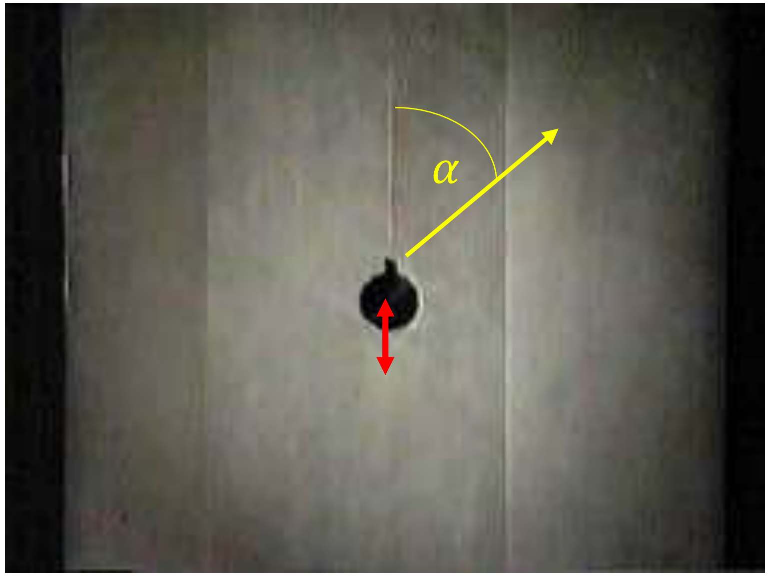
Visualisation mechanism:
Light deflection due to changes in index of refraction, due to perturbations of density-stratification

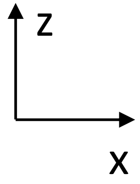


Görtler 1943
Sakai, Iizawa, Aramaki 1997

Changing forcing frequency, ω

N = constant





$$\boldsymbol{k} = k(\cos a, \sin a)$$

$$\omega = \omega(\alpha) = N \cos \alpha$$

$$a = \frac{p}{2} - Ð(\boldsymbol{g}, \boldsymbol{k})$$

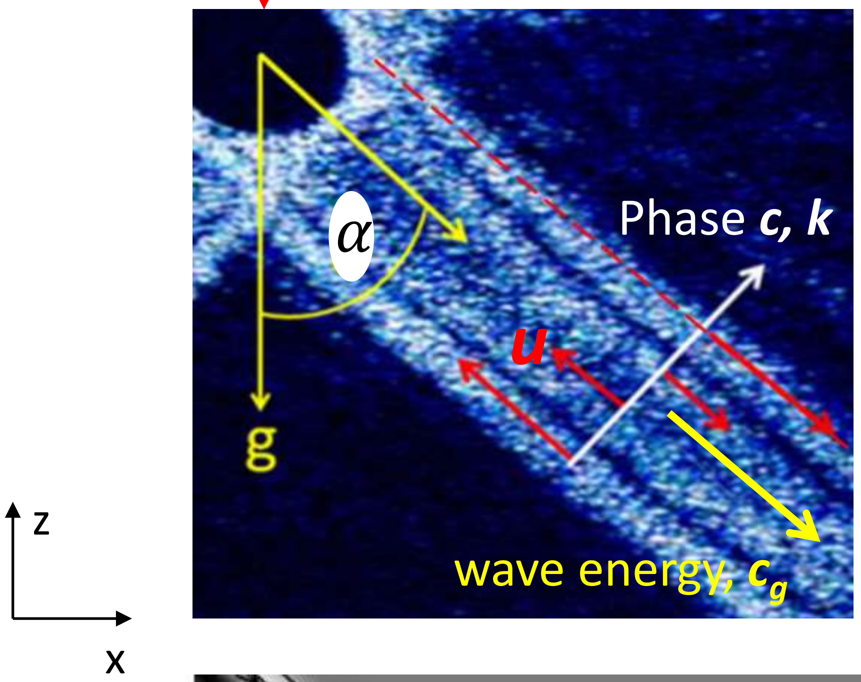
Frequency ⟶ angle, α

$$\boldsymbol{c}_g \wedge \boldsymbol{c}$$

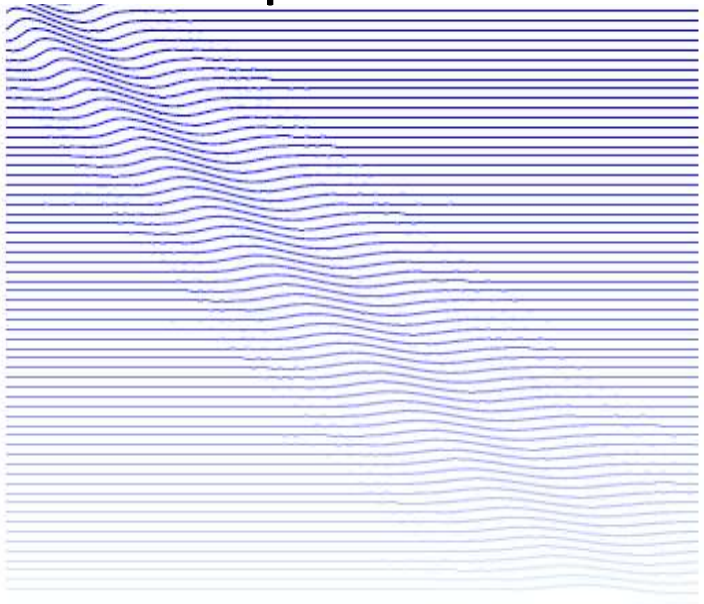
Görtler 1943
Sakai, Iizawa, Aramaki 1997

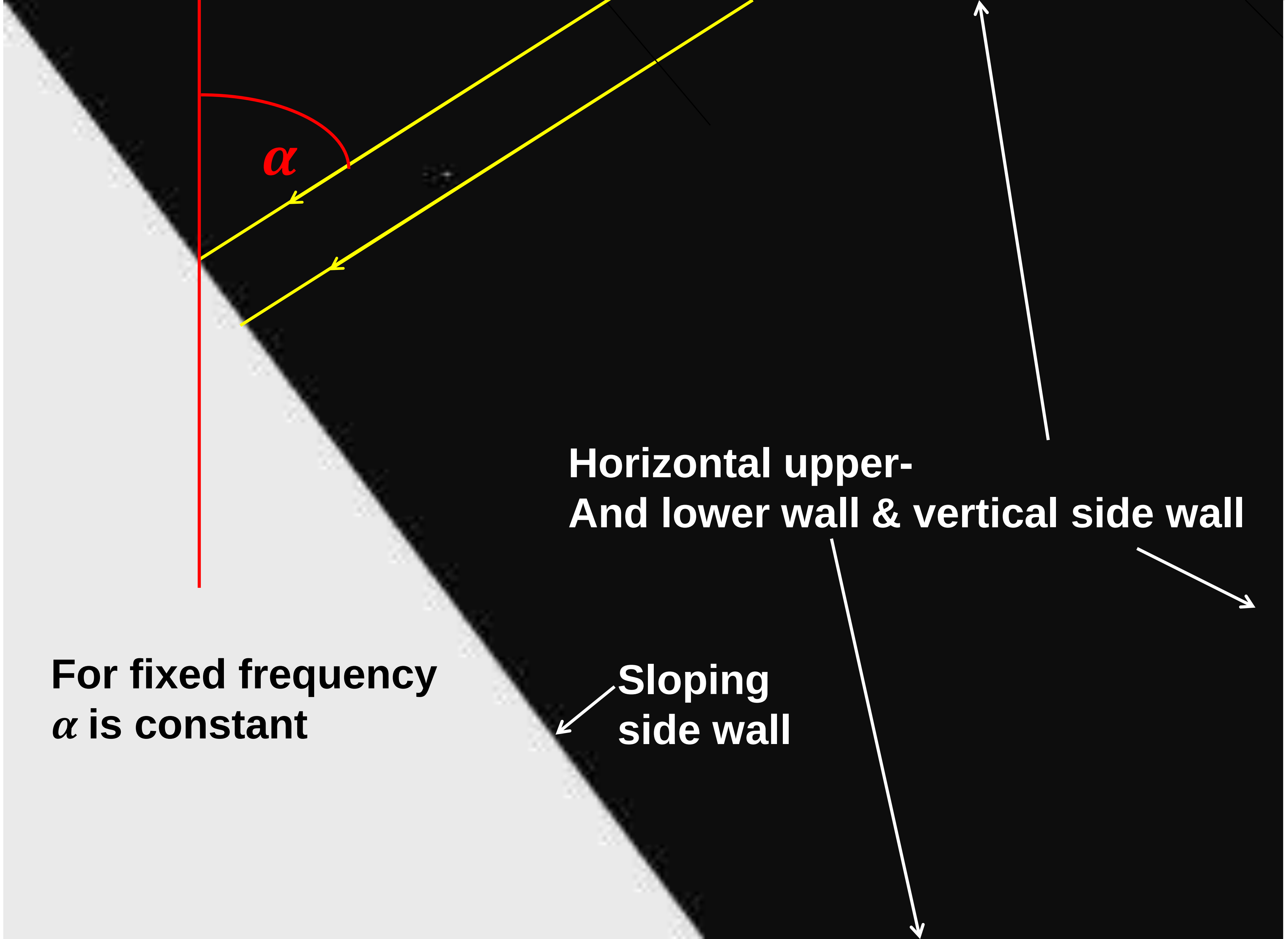
Waves in continuously-stratified fluid

Oscillate at frequency ω

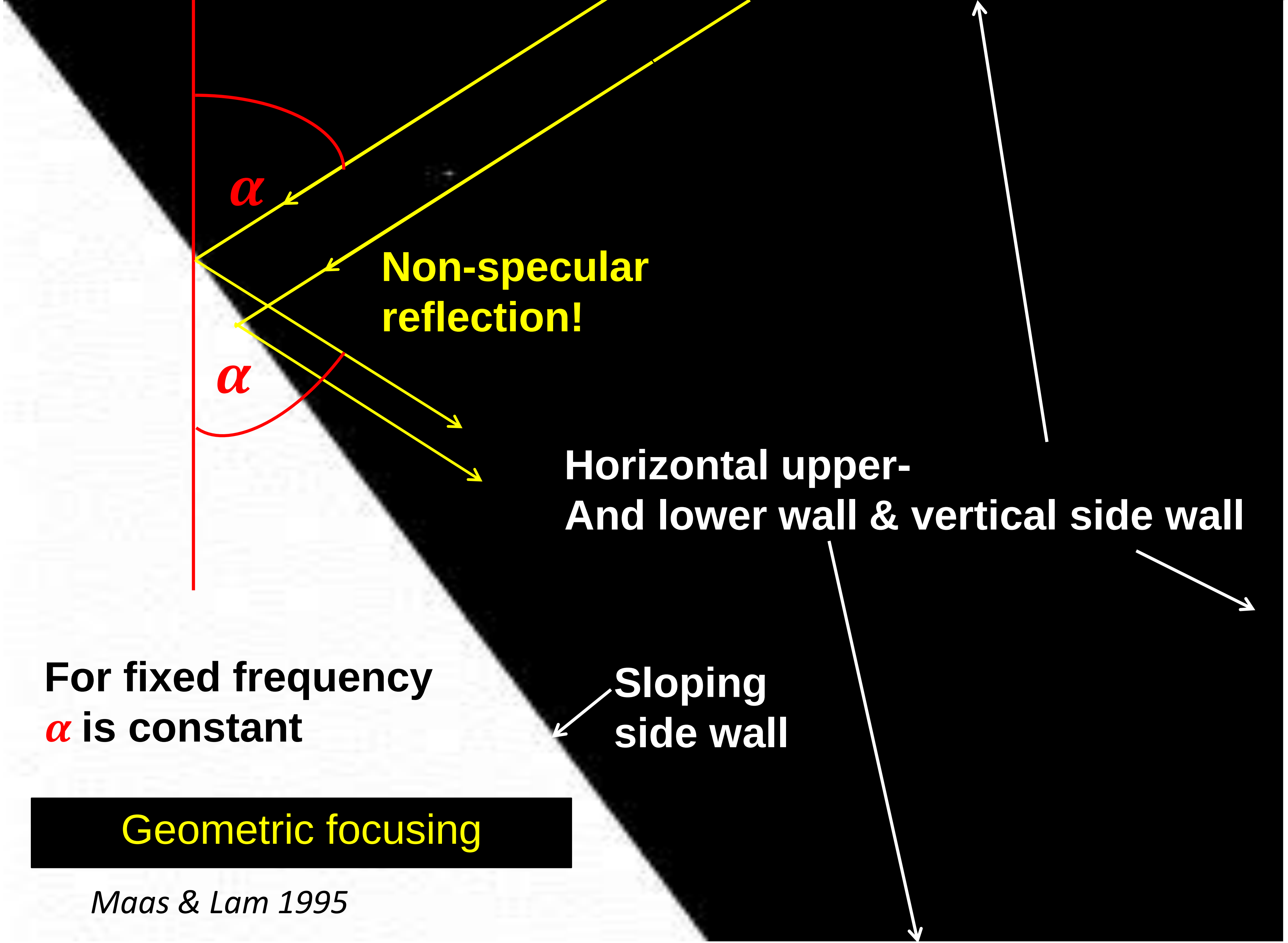


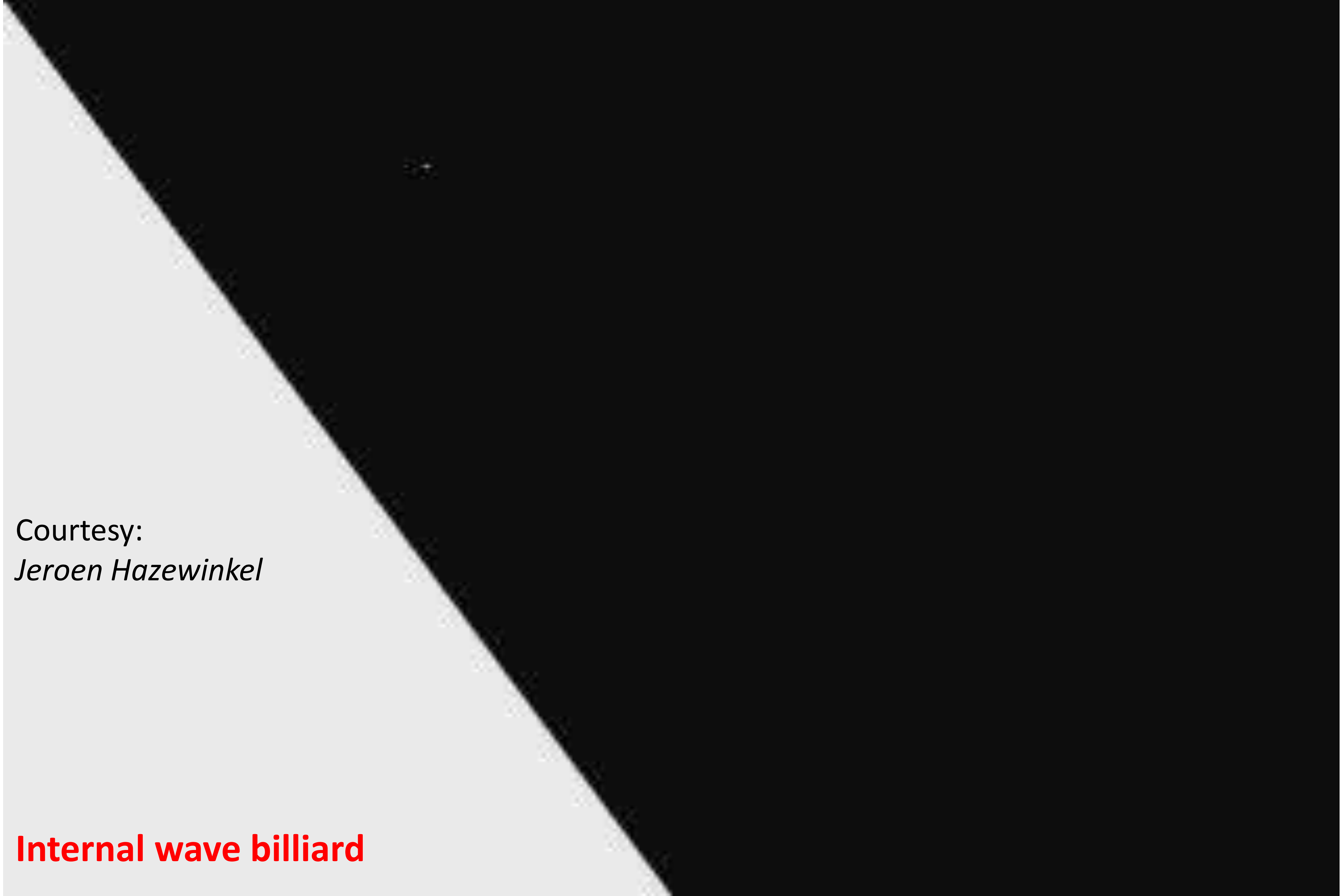
density contour displacements





α
Horizontal upper-
And lower wall & vertical side wall
For fixed frequency
α is constant
Sloping
side wall

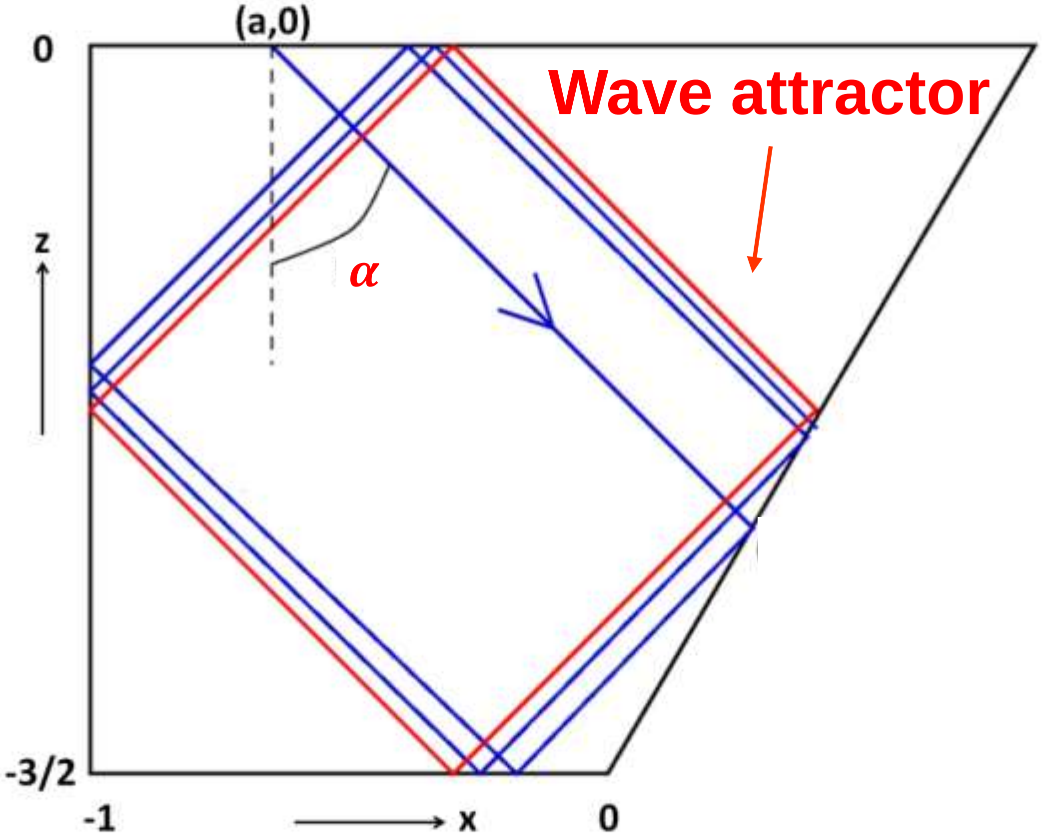

α
Non-specular
reflection!
α
Horizontal upper-
And lower wall & vertical side wall
For fixed frequency
α is constant
Sloping
side wall
Geometric focusing
Maas & Lam 1995

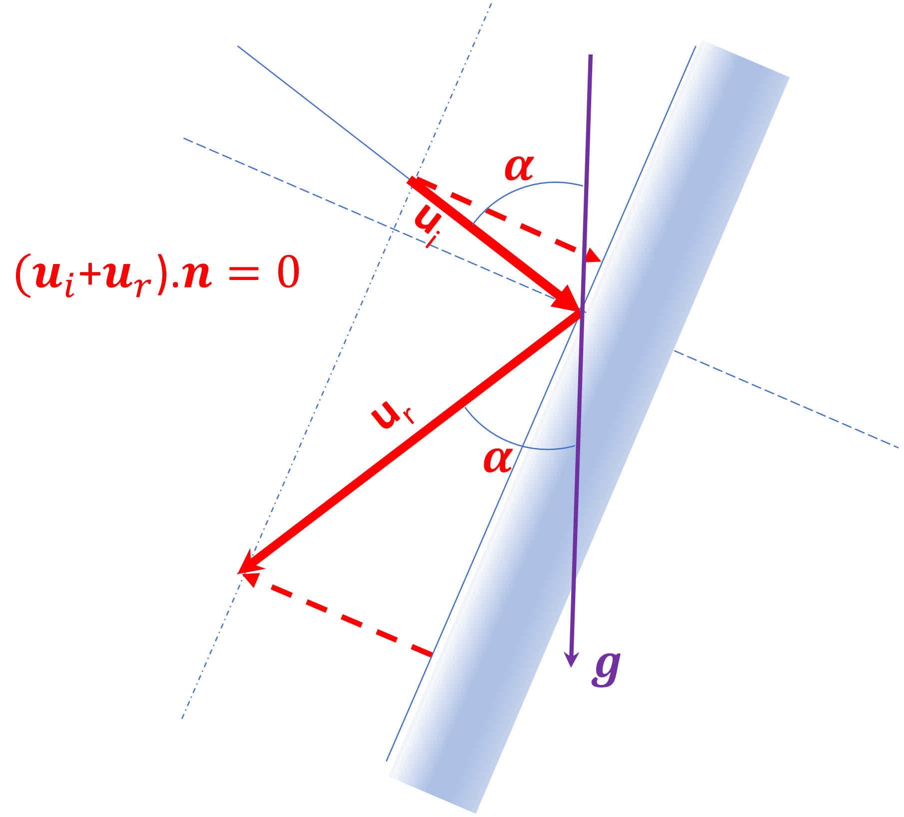


Courtesy:
Jeroen Hazewinkel

Internal wave billiard

Non-specular reflection
$\rightarrow amplification$

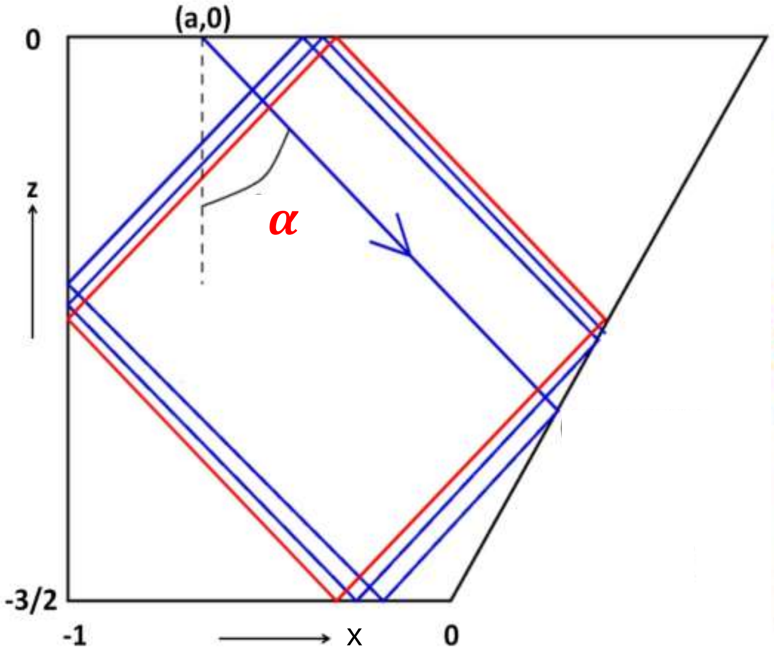


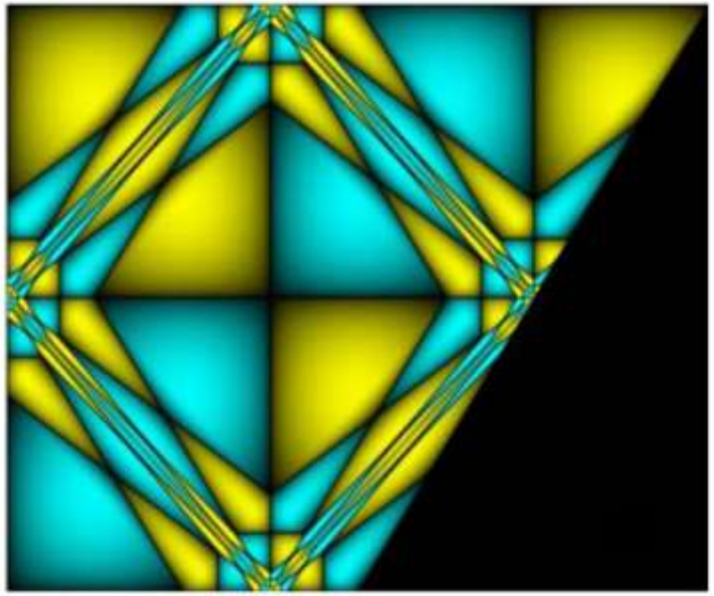


Indefinite focusing!

$$\nabla \bullet \boldsymbol{u} = 0 \quad \rightarrow \quad \boldsymbol{k} \bullet \boldsymbol{u} = 0$$
$$\boldsymbol{c} \bullet \boldsymbol{c}_g = 0 \quad \rightarrow \quad \boldsymbol{k} \bullet \boldsymbol{c}_g = 0$$
$$\boldsymbol{u} \| \boldsymbol{c}_g$$

Wave attractor properties





$$\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right)\psi=0$$

Unobservable streamfunction:

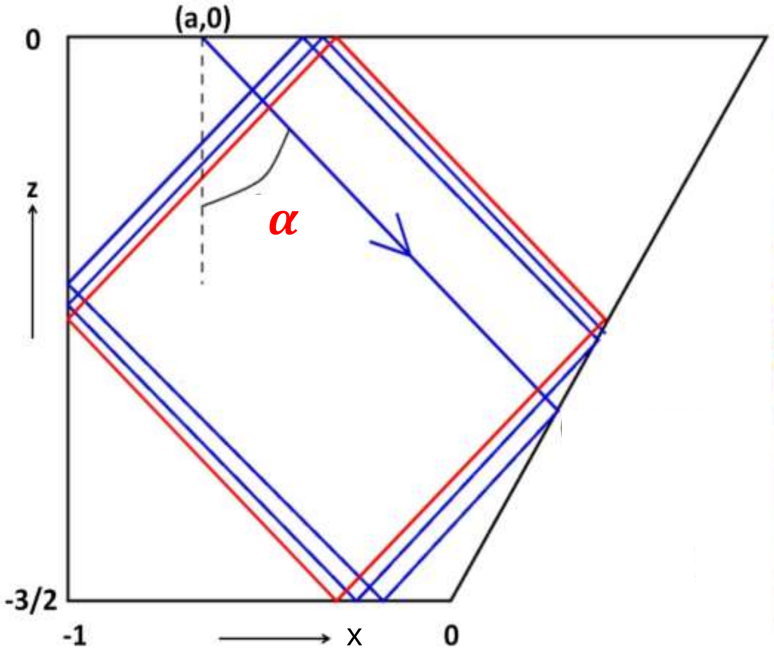
$$\psi(x,z)=f(x+z)-g(x-z)$$

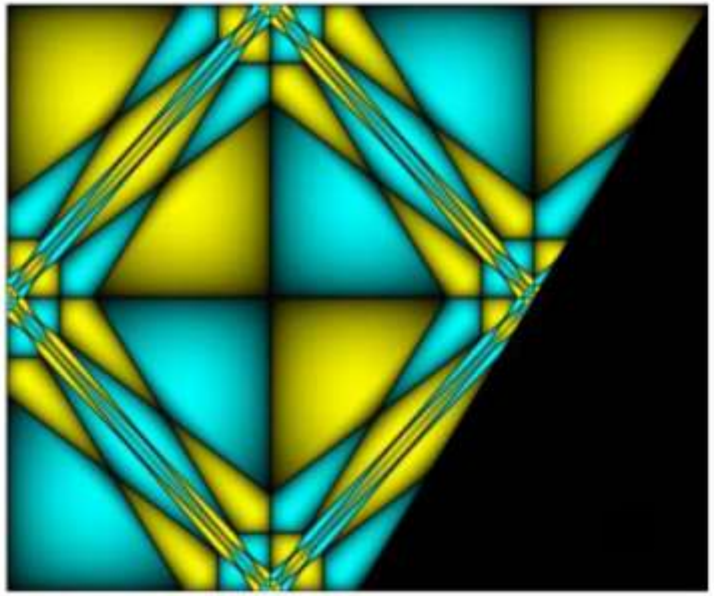
leads to amplified velocities (proportional to streamfunction derivatives) near focusing locations

$$\boldsymbol{u}=(u,w)=\left(\frac{\partial y}{\partial z},-\frac{\partial y}{\partial x}\right)=\left(\frac{\partial p}{\partial x},-\frac{\partial p}{\partial z}\right)$$

Multi-scale solutions of *linear* spatial wave equation,
using *nonlinear map* of boundary onto itself,
are selfsimilar in real space, parameter space and Fourier space.

Wave attractor properties





$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)\psi = 0$$

Unobservable streamfunction:

$$\psi(x, z) = f(x+z) - g(x-z)$$

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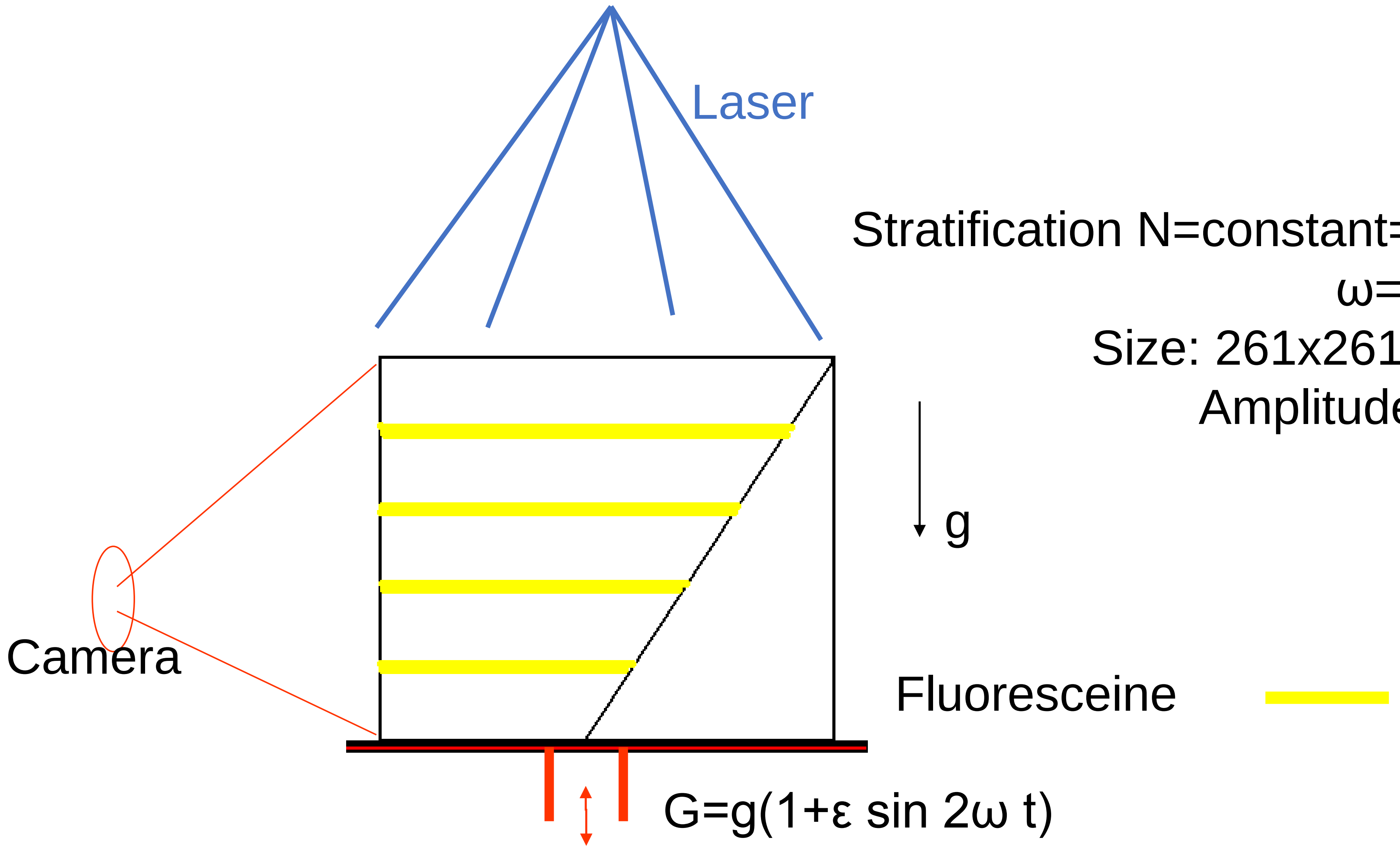
$$\boldsymbol{u} = (u, w) = \left(\frac{\partial y}{\partial z}, -\frac{\partial y}{\partial x}\right) = \left(\frac{\partial p}{\partial x}, -\frac{\partial p}{\partial z}\right)$$

Multi-scale solutions of *linear* spatial wave equation,
using *nonlinear map* of boundary onto itself,
are selfsimilar in real space, parameter space and Fourier space.

M.C. Escher

Wave attractor experiment



Stratification N=constant=1.89 s^{-1}

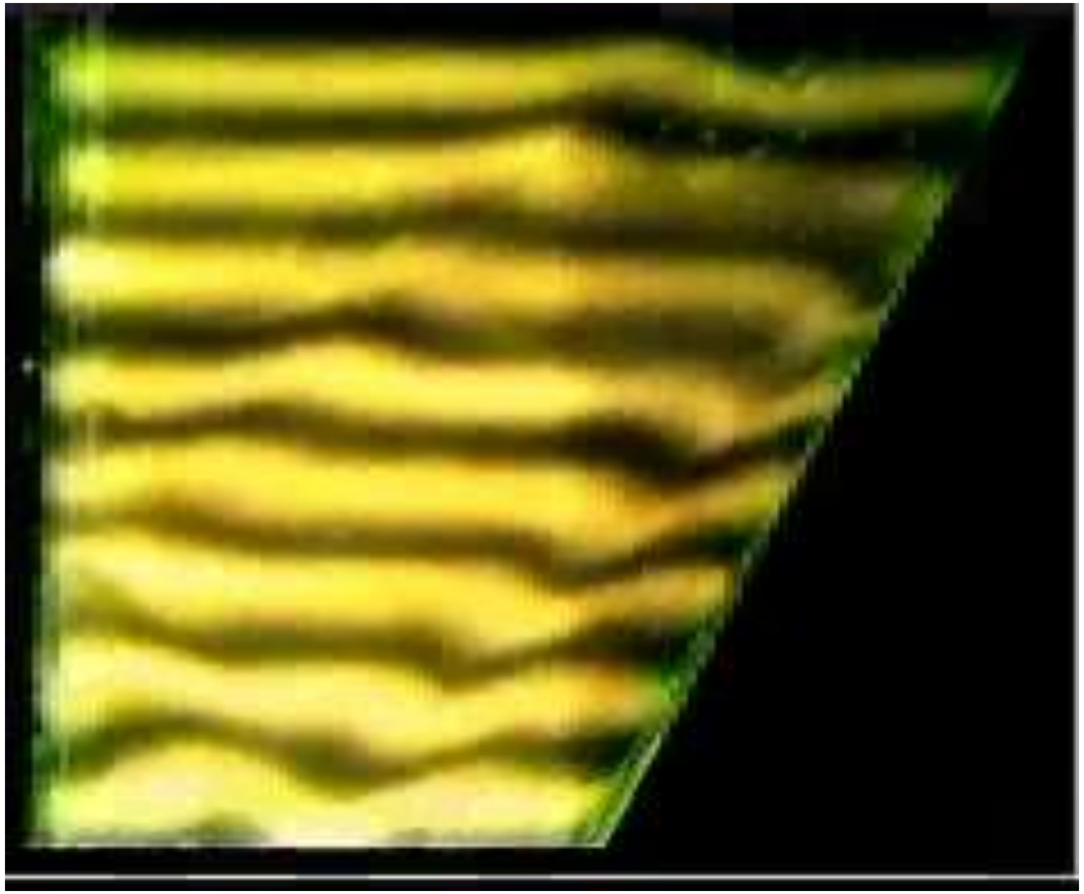
ω= 1.44 s^{-1}

Size: 261x261x96 mm

Amplitude :10 cm

Maas, Benielli, Sommeria & Lam 1997

Dye displacement



subtracting initial lines

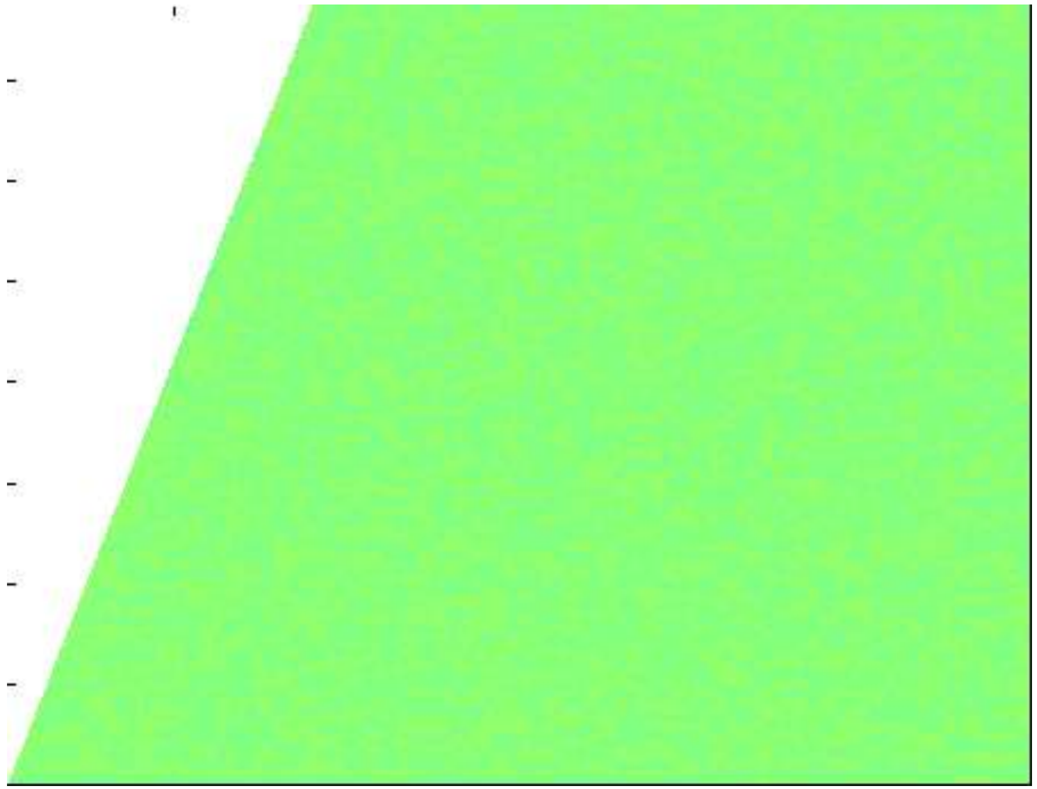
Oscillations start after ≈ 5 min ≈ 50 oscillation periods

Side view uniformly-stratified tank

Forcing by parametric excitation

Maas et al 1997

Shaking horizontally: growth phase

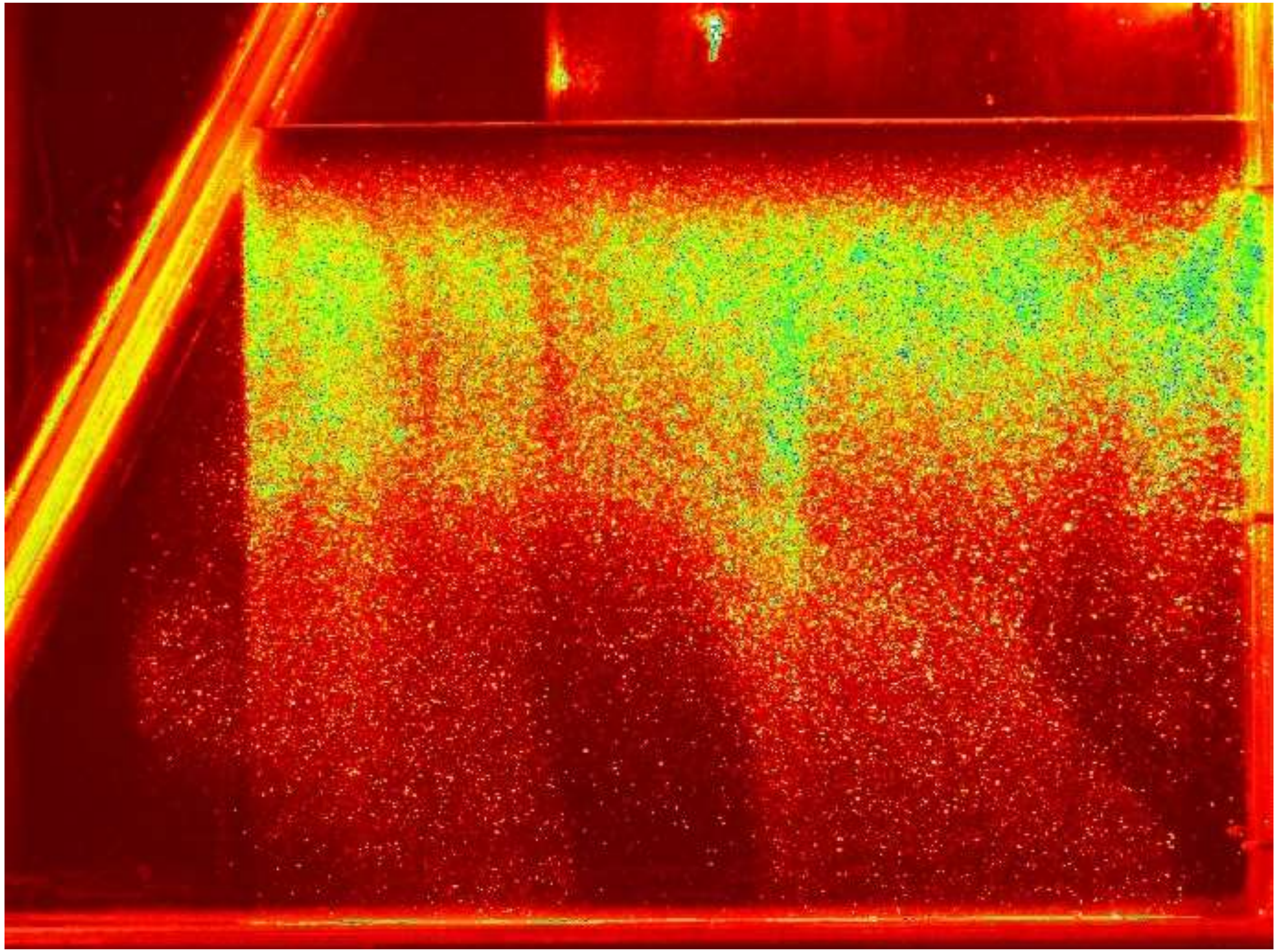


Side view of density perturbation gradient

Stroboscopic movie

Viscous saturation: *Hazewinkel, v Breevoort, Dalziel & M. 2008*
Triadic instability attractor: *Scolan et al 2013, Brouzet et al 2016*

Particle transport



Courtesy: Jeroen Hazewinkel

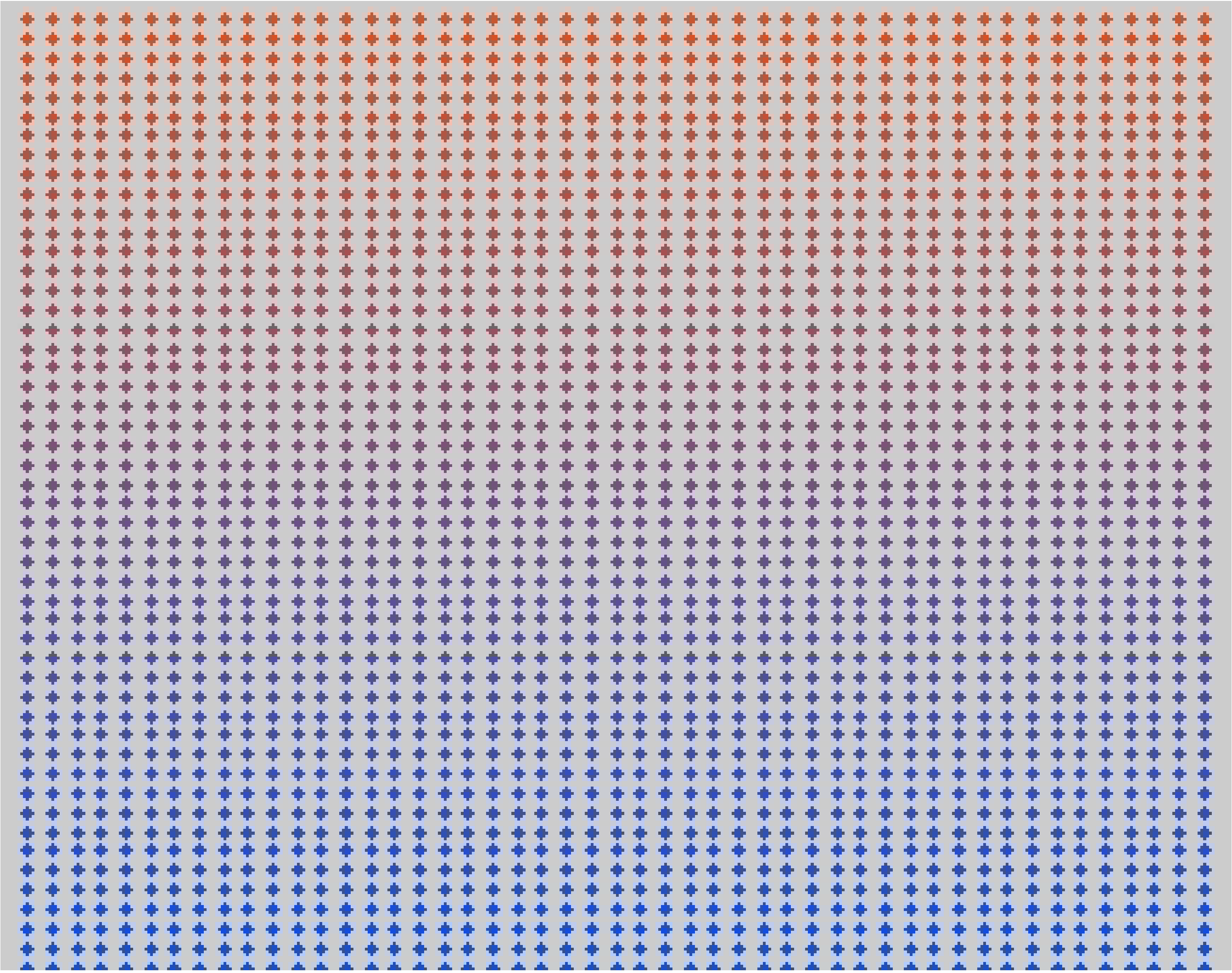
3 periods of oscillations, followed by stroboscopic view over many periods

Displacement of particles provides **u**(x,y,t)

Integrate kinematic equations

d**x**/dt=**u**(x,y,t)

trajectories virtual particles

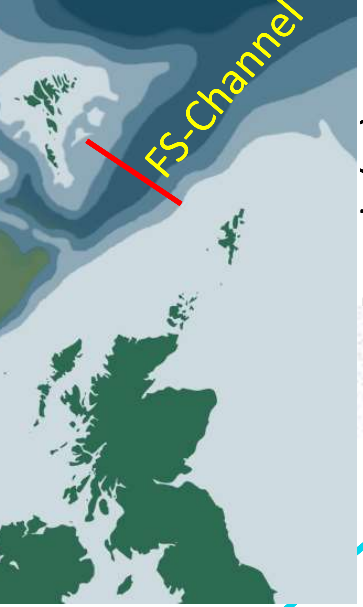


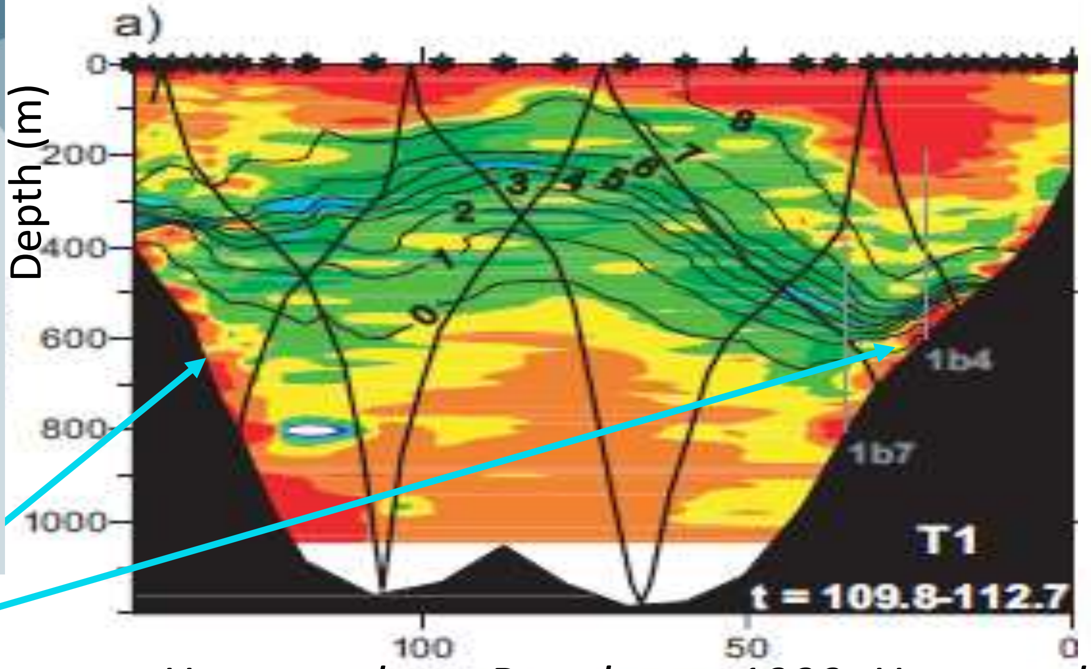
See also:
Beckebanze, Brouzet, Sibgatullin, Maas 2017

Courtesy: Jeroen Hazewinkel

... wave attractors also attract particles!

Wave attractor in Faroe Shetland channel?

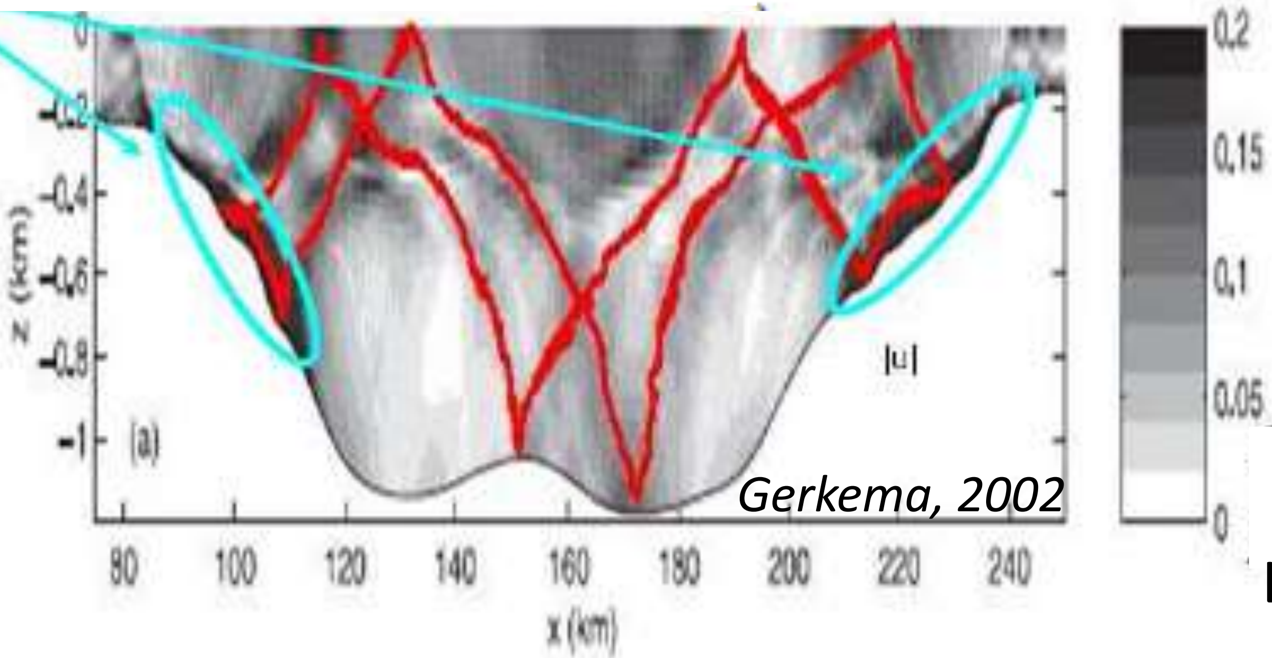




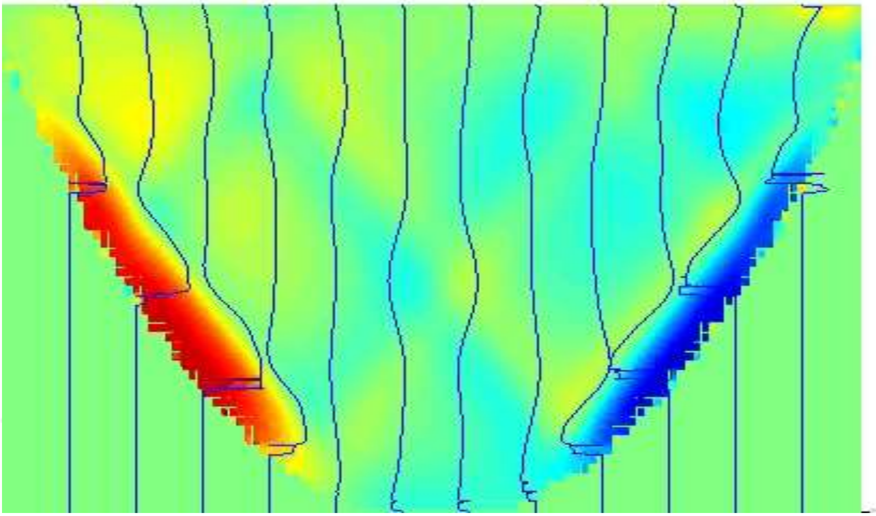
Field observations: Isotherms (°C)
Vertical diffusivity: (green: low, red: high)
Model : (Curved) internal tidal rays

Enhanced mixing and flows

van Haren and van Raaphorst, 1999; Hosegood et al 2005



Gerkema, 2002



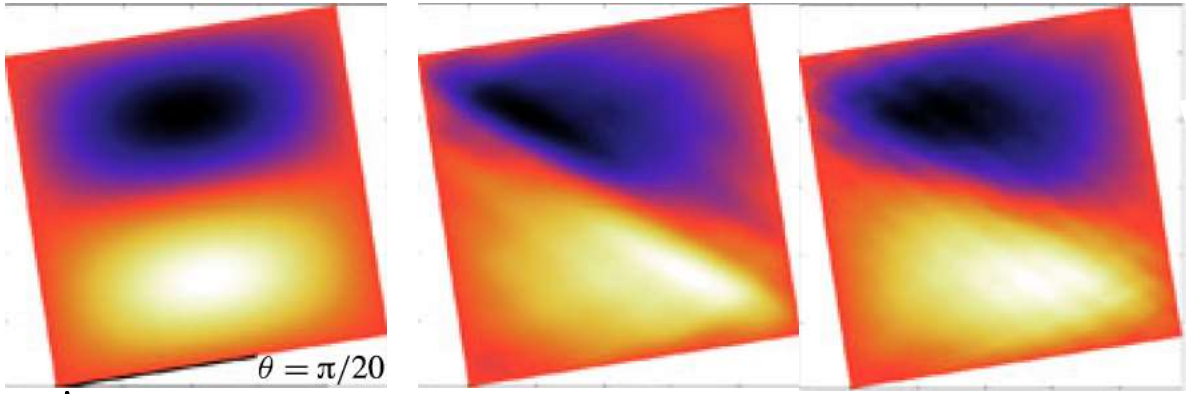
Lab observation *density perturbation*

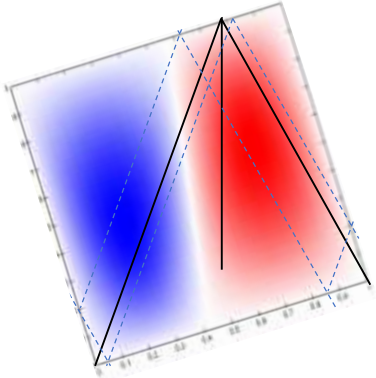
Courtesy: Jeroen Hazewinkel

Initial Value Problem uniformly-stratified fluid

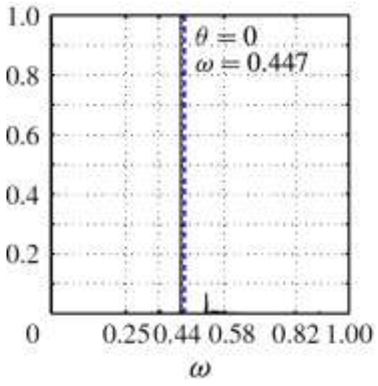
Frequency spectrum

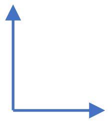
Initial
stream function
disturbance





NO attractor





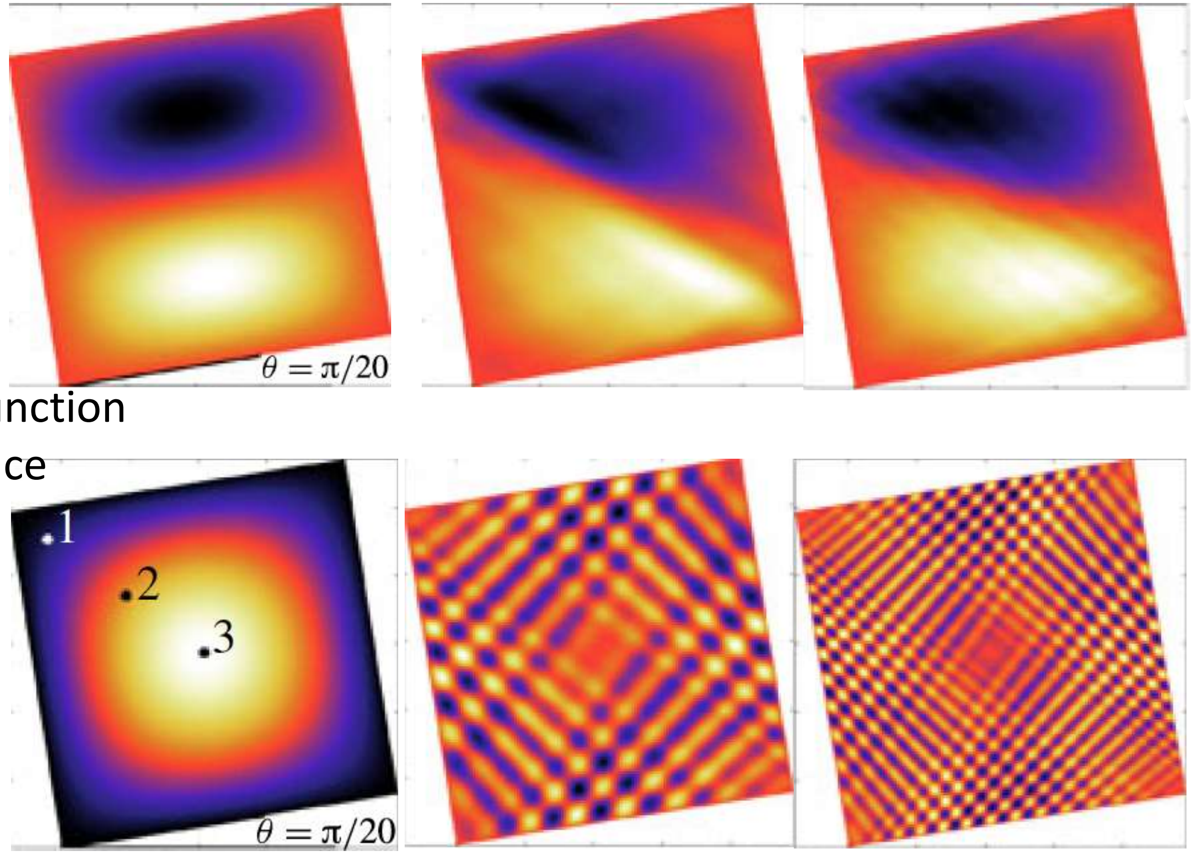
t=0 t=100 t=200

Streamfunction: structure-preserving numerical method

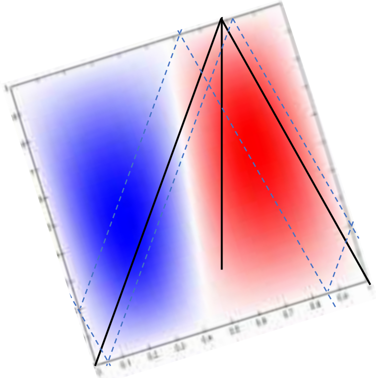
Bajars, Frank, Maas 2013

Initial Value Problem uniformly-stratified fluid

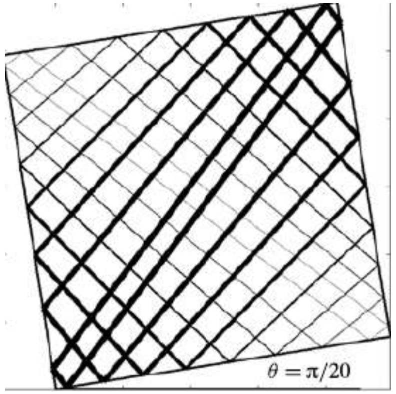
Frequency spectrum

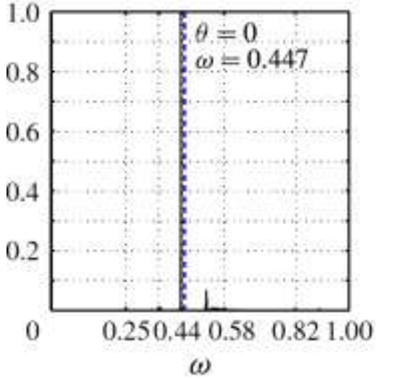


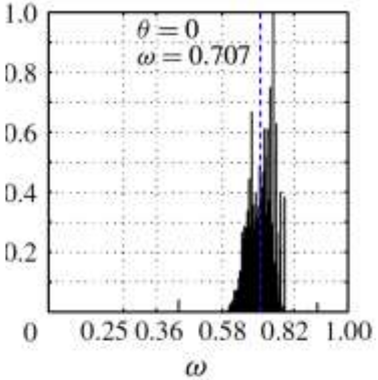
Initial stream function disturbance



NO attractor







z

x

t=0 t=100 t=200

Range of (1,1) attractors

Broad-band response in (1,1) - band

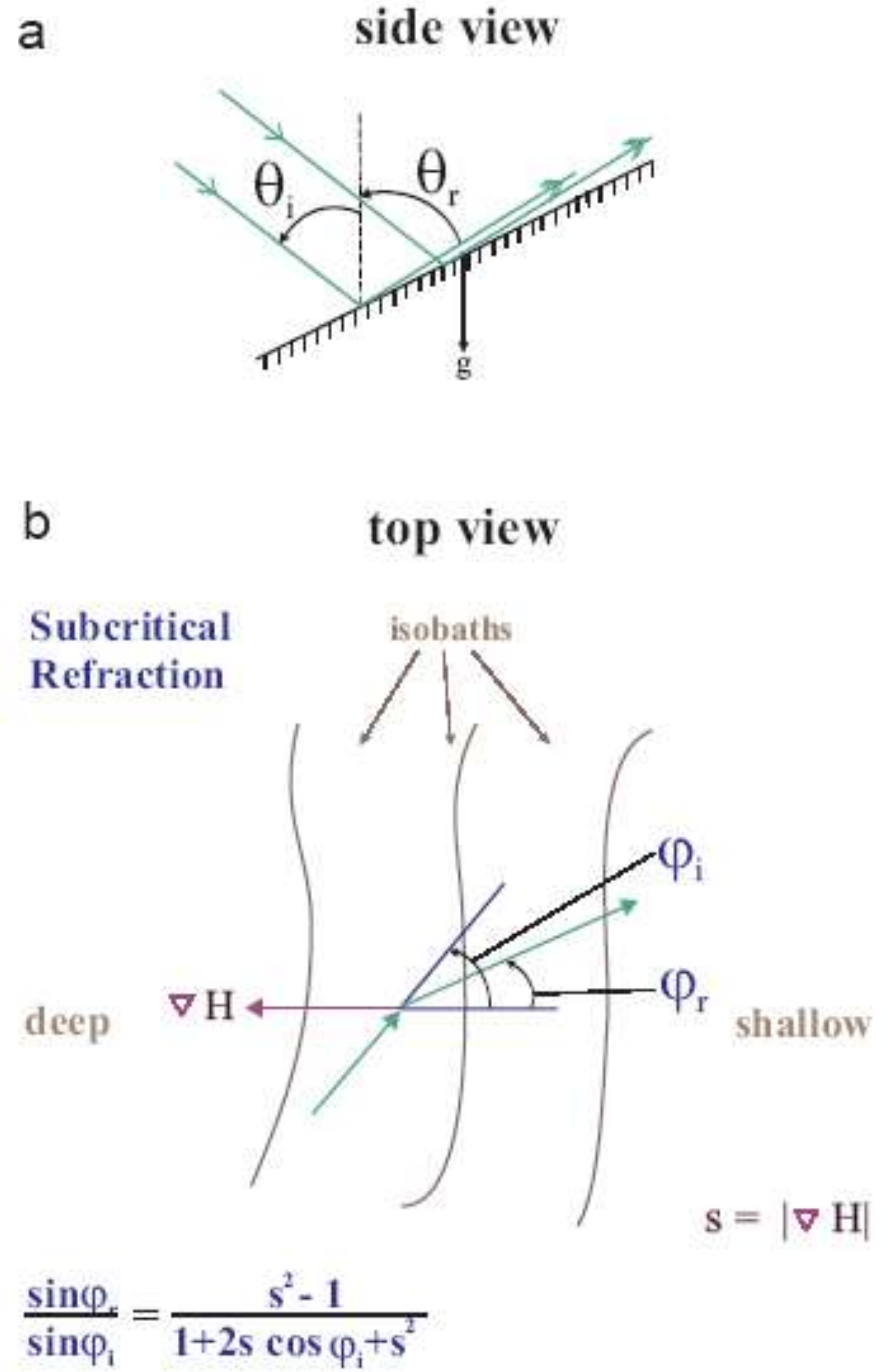
Streamfunction: structure-preserving numerical method

Bajars, Frank, Maas 2013

Three-dimensional effects

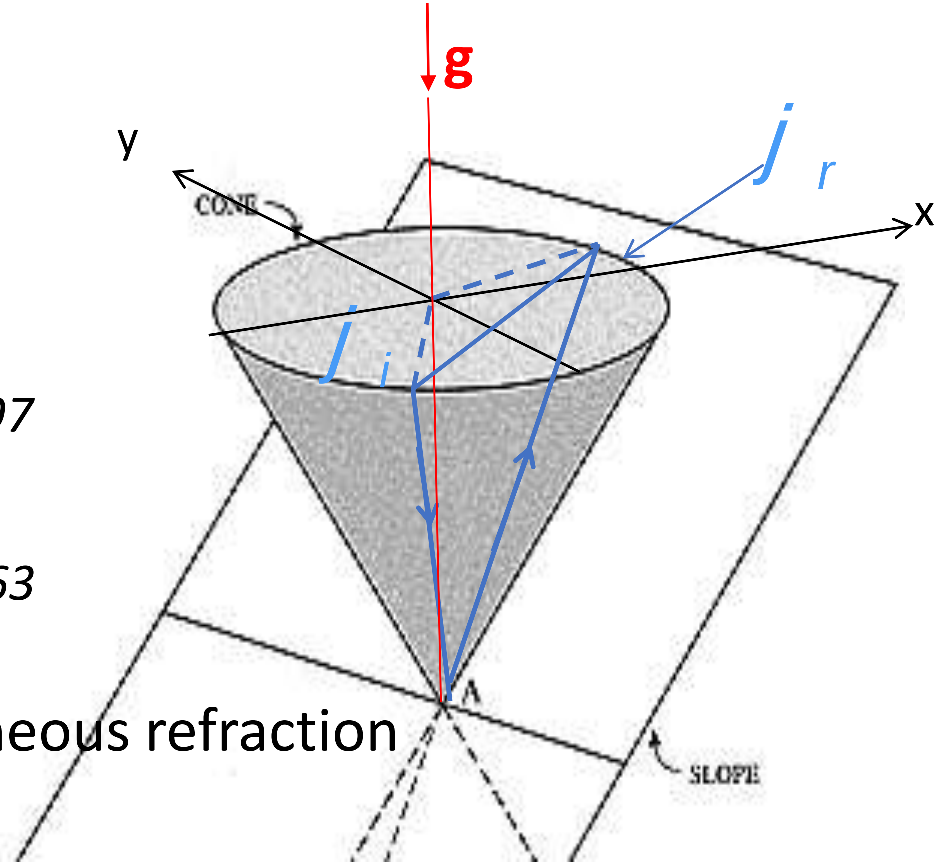
Reflection of obliquely incident ray

Poincaré-Sobolev equation: $P_{xx} + P_{yy} - P_{zz} = 0$



Thorpe 1997

Phillips 1963



Instantaneous refraction

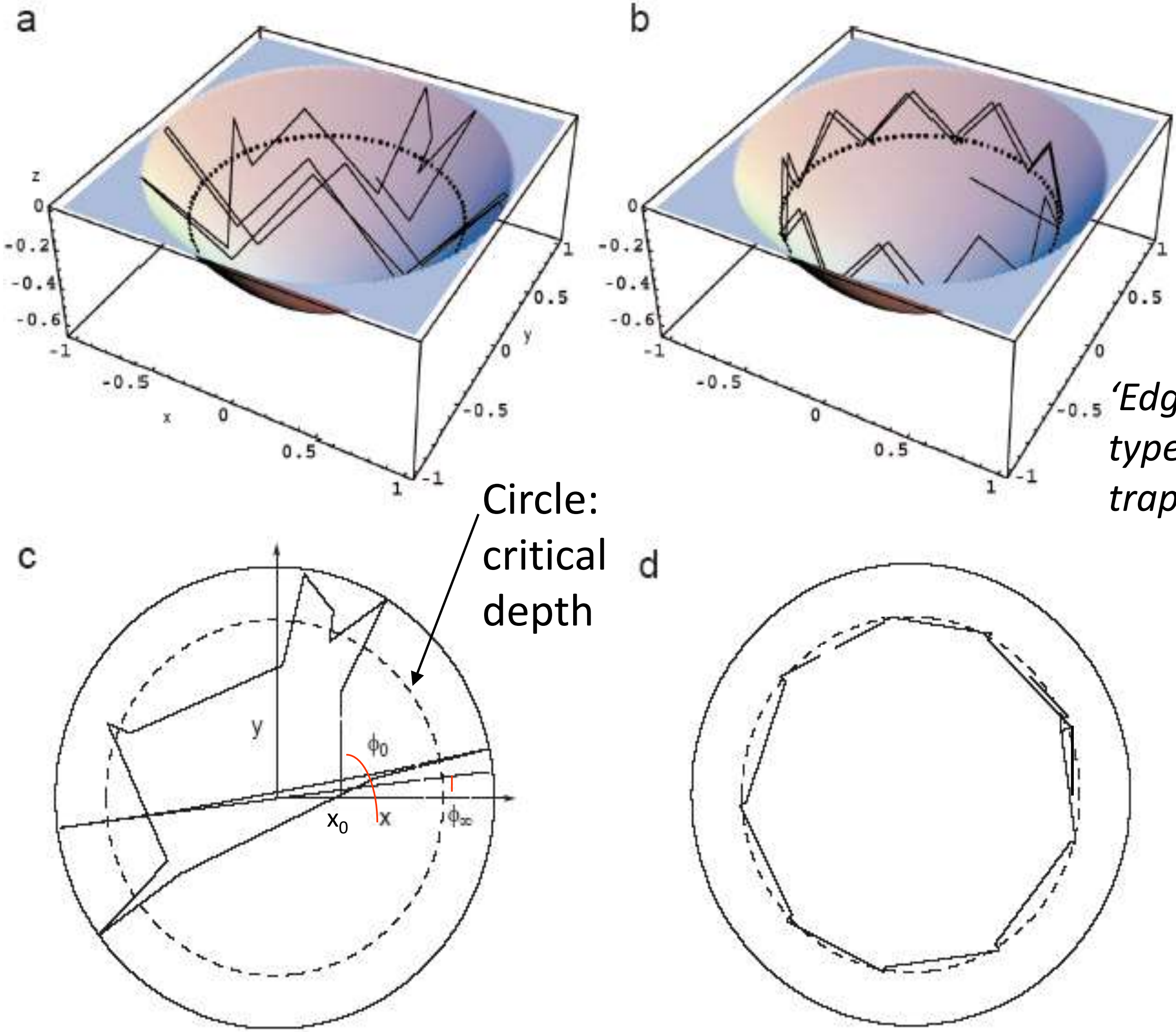
Ray tracing in uniformly-stratified paraboloid

N=const

3D view

Focusing on wave attractor

'Edge wave' type trapping

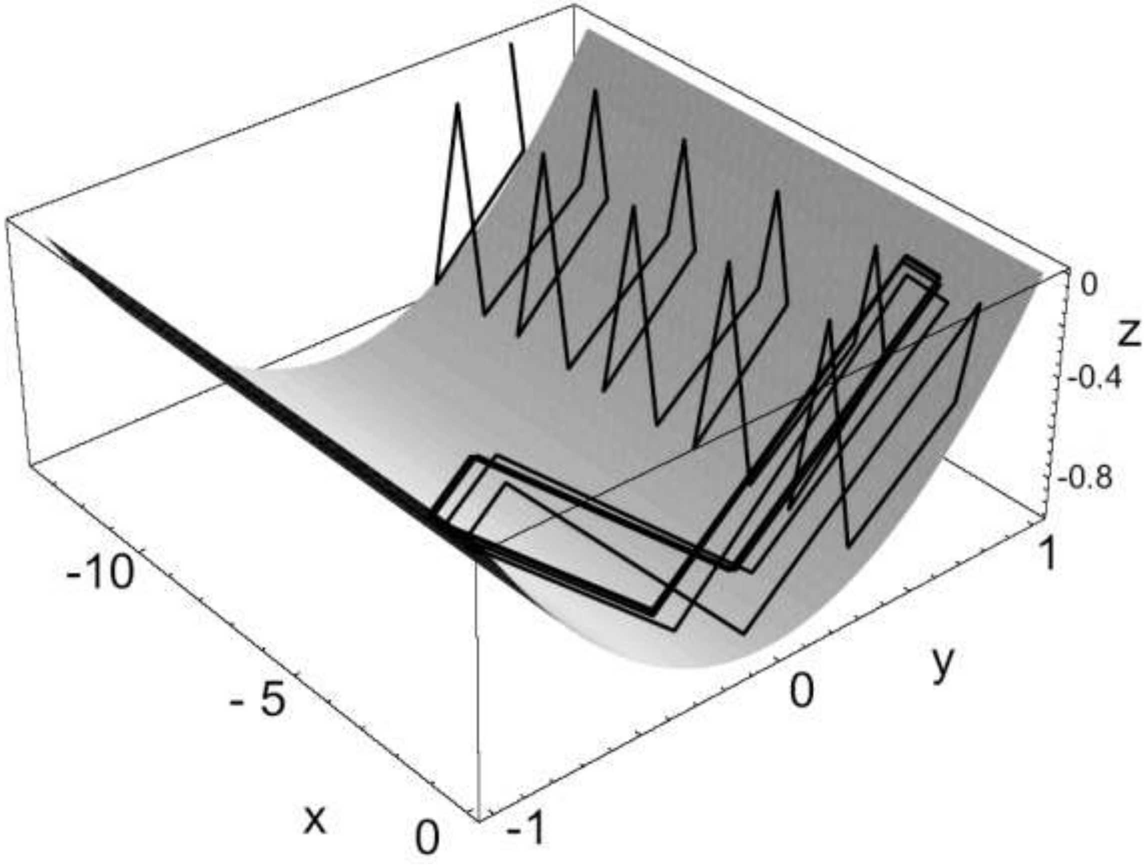


Circle: critical depth

Top view

Maas 2005

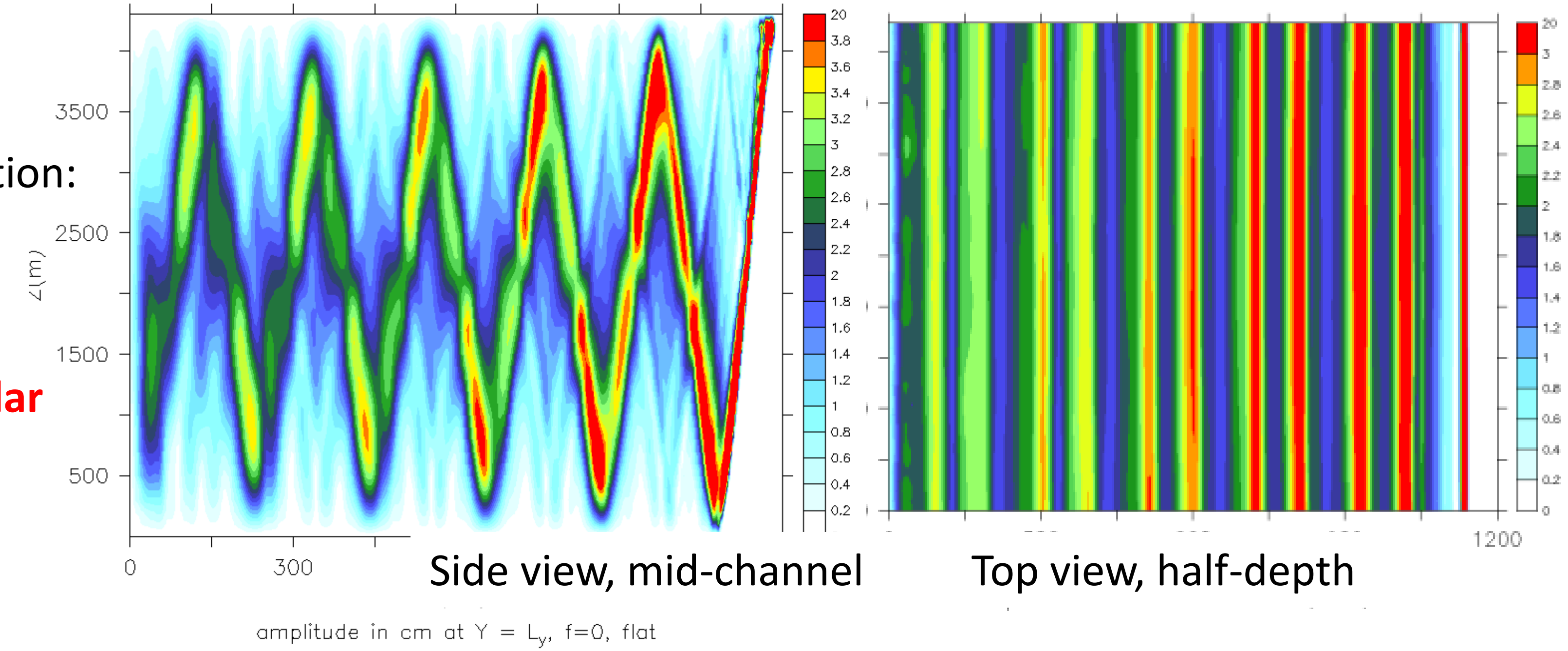
Internal wave ray paths in uniformly-stratified parabolic channel



Internal tide generation in MICOM - dependence of cross-channel geometry

Cross-section:

Rectangular



Side view, mid-channel

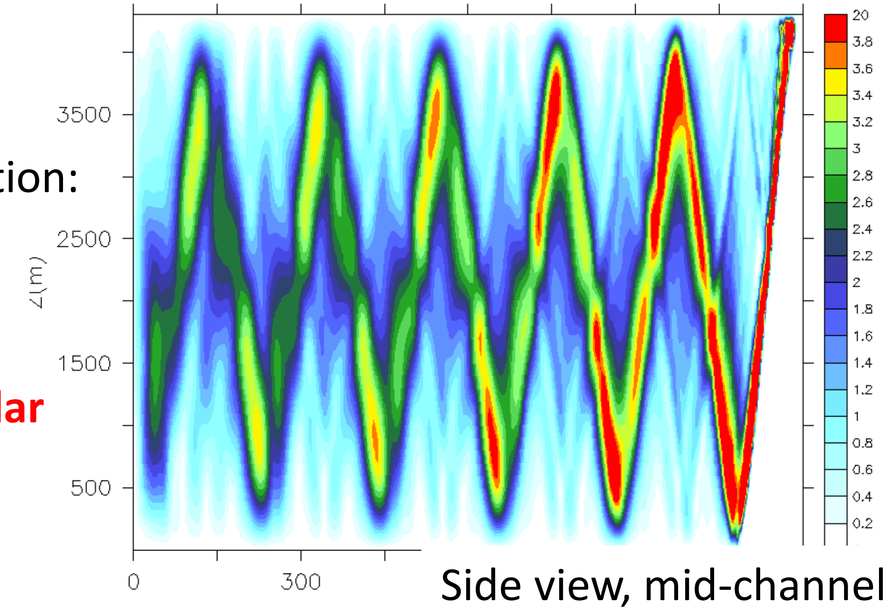
Top view, half-depth

Drijfhout & Maas 2007

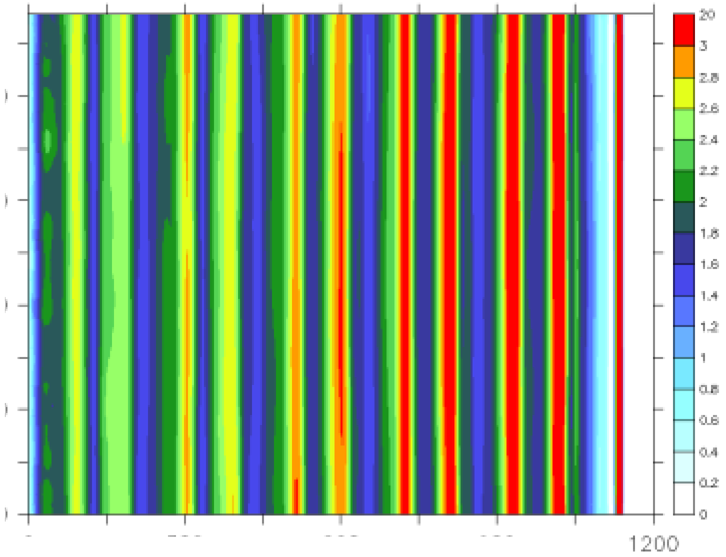
Internal tide generation in MICOM - dependence of cross-channel geometry

Cross-section:

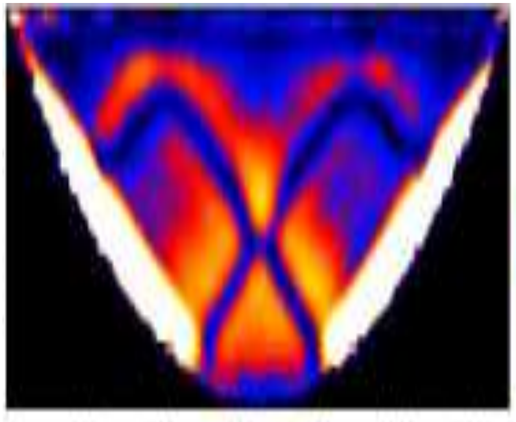
Rectangular



Side view, mid-channel

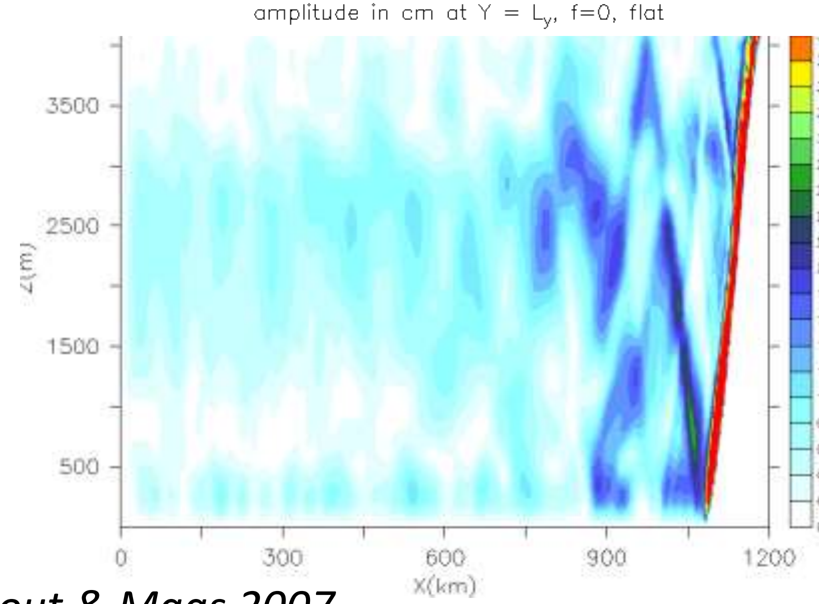


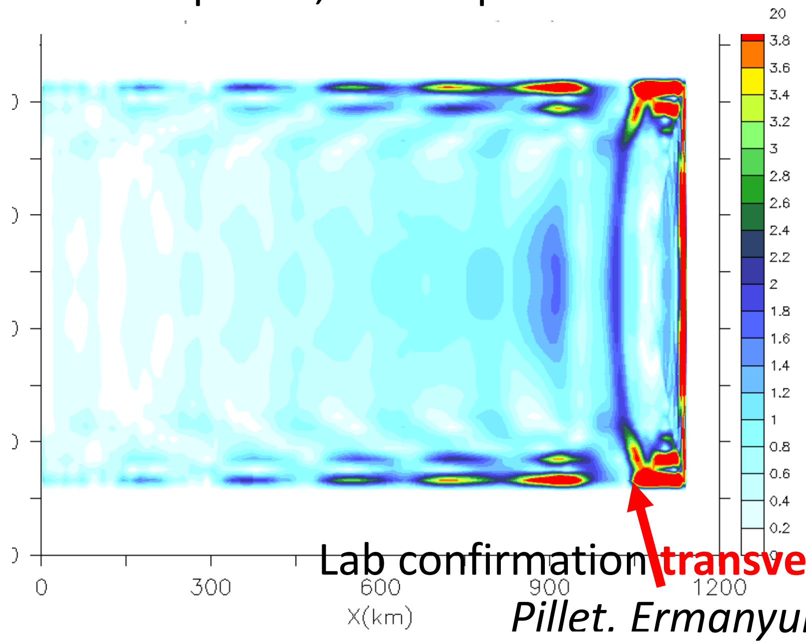
Top view, half-depth



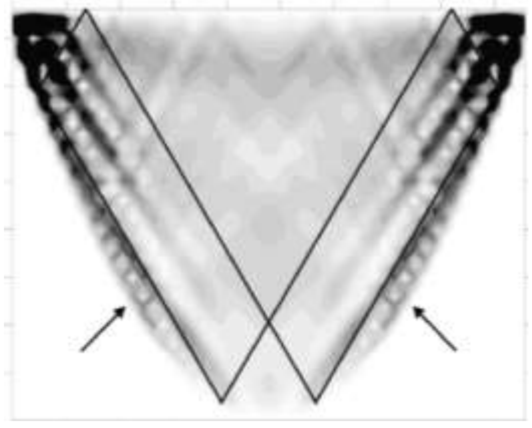
Laboratory Exp Amplitude

Parabolic





Transverse view



Lab confirmation **transverse trapping**:

Pillet. Ermanyuk, Maas, Sibgatullin, Dauxois (2018)

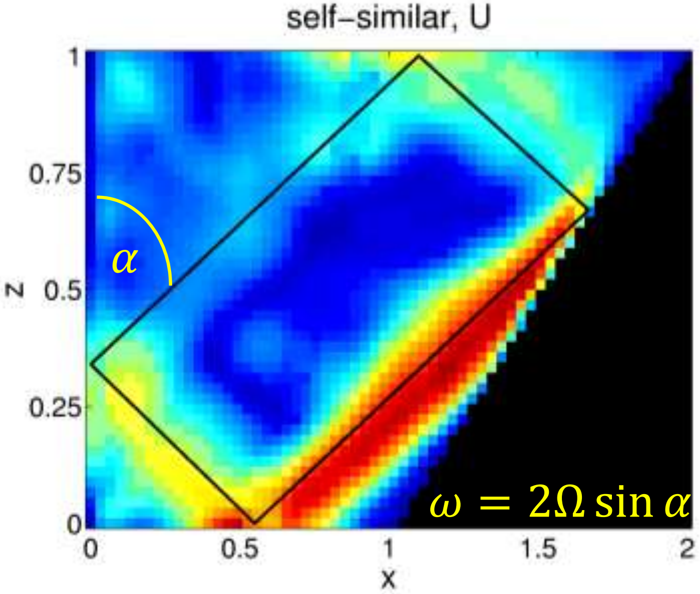
Drijfhout & Maas 2007

Wave attractors in other anisotropic media?

Homogeneous, rotating fluid experiments

In Geophysical and Astrophysical media
- Rotating fluids (inertial wave)
- Plasma's subject to magnetic field (electron-cyclotron waves)

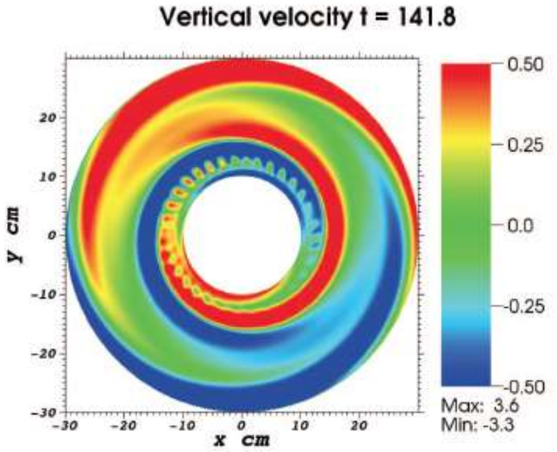
Lab experiment in trapezoidal channel, forcing by slight modulation of angular speed

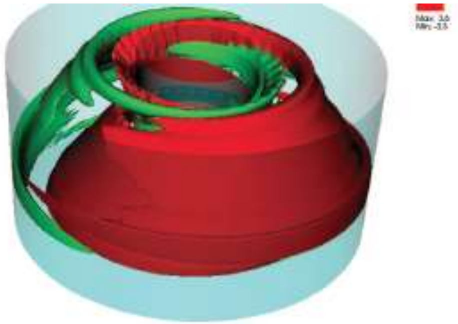


Velocity magnitude

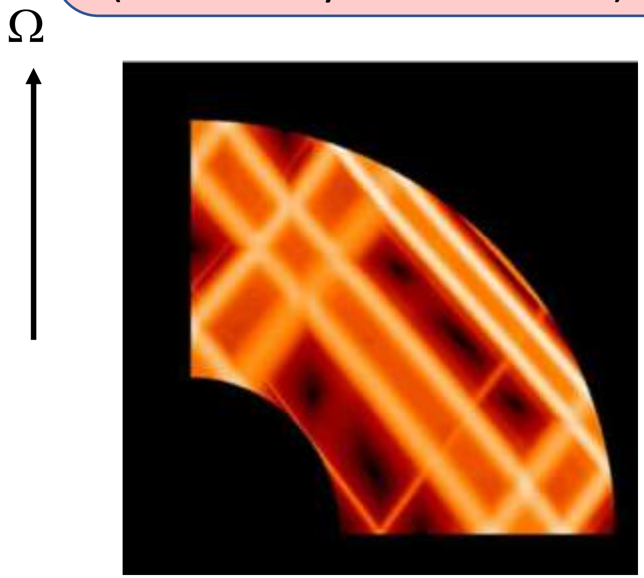
Maas 2001, Manders & Maas 2003

Forcing by nutation of lid





NEK5000 computation



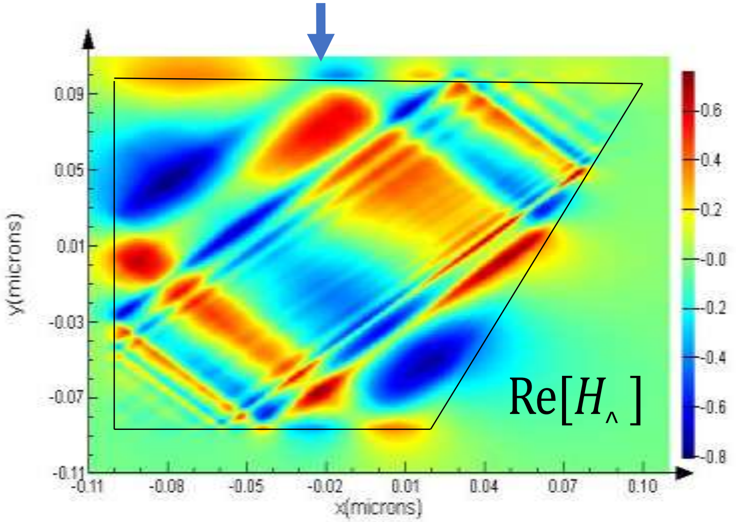
Numerics: Planetary & Stellar interiors
Rieutord 2009

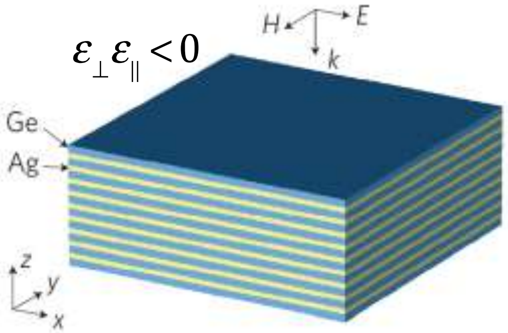
Sibgatullin, Ermanyuk, Maas, Xiulin, Dauxois 2017

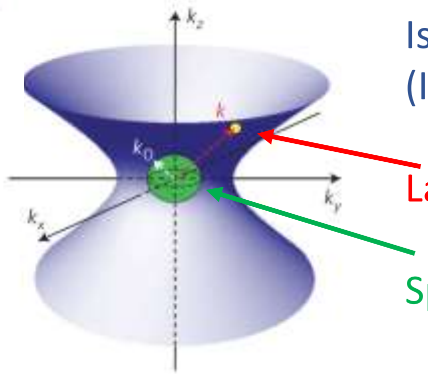
Wave attractors in other anisotropic media?

In EM and acoustic waves
- Hyperbolic Metamaterial (light at nanoscale)
- Acoustic Metamaterials?

Dipolar source





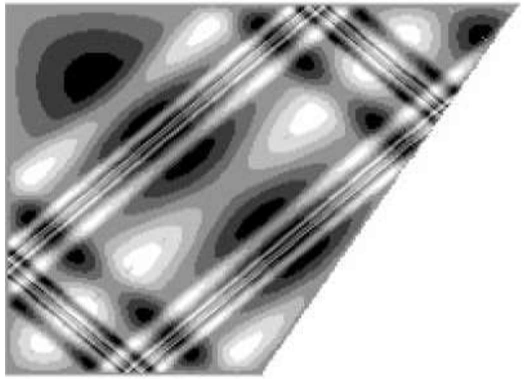


Hyperboloidal Iso-Frequency Contour (IFC) in HMM

Large k, small wavelength $w = w(a)$

Spherical IFC in vacuum $w = w(|\mathbf{k}|)$

Natural, continuous Hyperbolic Metamaterial



$H = H_\perp$ acts as stream function ψ

Displacement (induction) grows without bound… $\boldsymbol{D} = \frac{i}{w}\left(\frac{\partial H}{\partial z}, -\frac{\partial H}{\partial x}\right)$

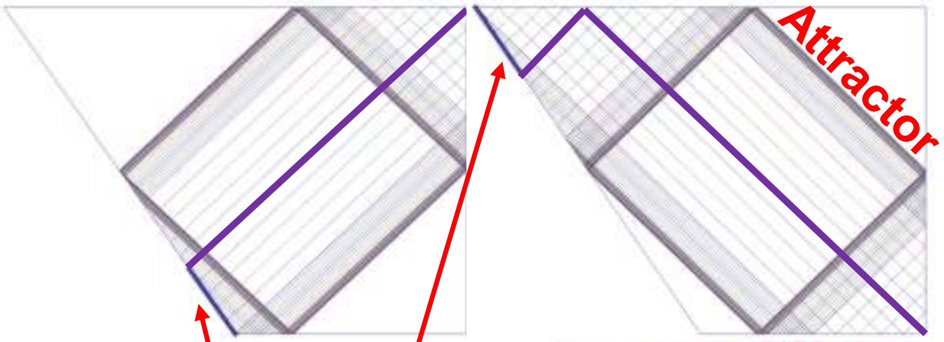
Cellular pattern in ‘outer’ region

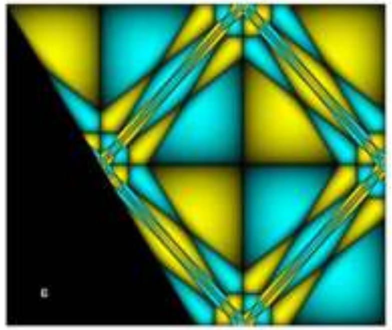
Ruben Maas 2016

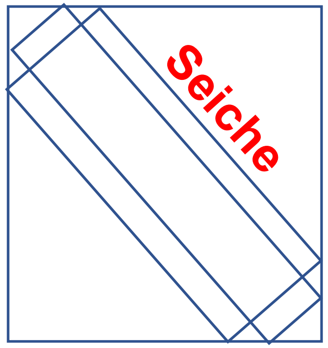
Summary: Basin shape matters!

$\omega = N \cos \alpha$

Courtesy: Anna Rabitti

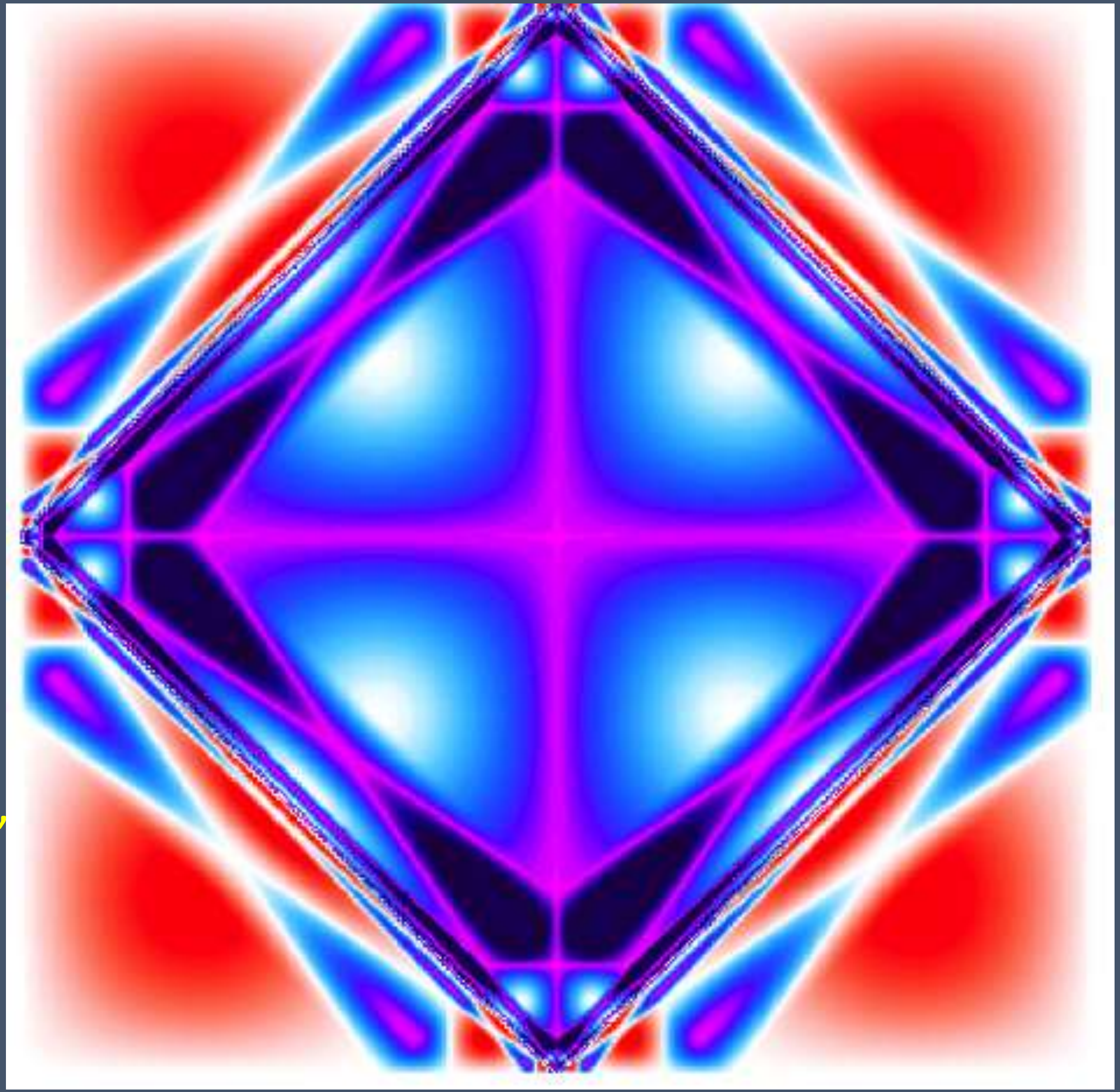






Anisotropic media support waves that focus onto wave attractors: mixing locations, also attracting particles

Спасибо



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